Math 255A' Lecture 16 Notes

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1 Adjoints and Hermitian Operators on Hilbert Spaces

Today's lecture was given by a guest lecturer, Professor Sorin Popa.

1.1 Sesquilinear forms and adjoints

If $T \in \mathcal{B}(X, Y)$, we have the adjoint operator $T^* \in \mathcal{B}(Y^*, X^*)$. If H, K are Hilbert spaces, then $H^* \cong \overline{H}$, the conjugate of H (i.e. H itself). So if $T \in \mathcal{B}(H, K)$, we get $T^* \in \mathcal{B}(K, H)$.

Definition 1.1. A **sesquilinear form** is a function $u : H \times K \to \mathbb{C}$ which is linear in the first variable, antilinear in the second variable, and bounded (as a bilinear map): $|u(\xi, \eta)| \leq C\|\xi\|_H \|\eta\|_K$ for all $\xi \in H$ and $\eta \in K$.

Example 1.1. Let $A \in \mathcal{B}(H, K)$ and $B \in \mathcal{B}(K, H)$. Then $u_A(\xi, eta) = \langle A\xi, \eta\rangle_K$ and $u_B(\xi, \eta) = \langle \xi, B\eta\rangle_H$ are sesquilinear.

Theorem 1.1. *Let H, K be Hilbert spaces. If $u : H \times K \to \mathbb{C}$ is sesquilinear and bounded by C, then there exist unique $A \in \mathcal{B}(H, K)$ such that $u = u_A = u_B$ with $\|A\|, \|B\| \leq K$.*

Remark 1.1. In fact, $\|u\| = \|A\| = \|B\|$.

Proof. For each $\xi \in H$, let $L_\xi : K \to \mathbb{C}$ with $L_\xi(\eta) = \overline{u(\xi, \eta)}$. This is linear, and $|L_\xi(\eta)| \leq C\|\xi\|\|\eta\| =: C_\xi\|\eta\|$ for all η, so $L_\xi \in K^*$. By Riesz representation, there is an $f \in K$ with $\|f\| \leq C\|\xi\|$ such that $L_\xi(\eta) = \langle \eta, f\rangle$. Thus, $A : H \to K$ defined by $A(\xi) = f$ is linear: $A(\alpha_1\xi_1 + \alpha_2\xi_2) = \alpha_1 A(\xi_1) + \alpha_2 A(\xi_2)$ by the uniqueness in the Riesz representation theorem. We also have $\|A(\xi)\| \leq C\|\xi\|$, so A is bounded. □

Definition 1.2. If $A \in \mathcal{B}(H, K)$, the unique $B \in \mathcal{B}(K, H)$ that satisfies $u_A(\xi, \eta) = \langle A\xi, \eta\rangle_K = u_B(\xi, \eta) = \langle \xi, B\eta\rangle_H$ is called the **adjoint** of A (denoted A^*).

Proposition 1.1. *$u \in \mathcal{B}(H, K)$ is an isomorphism of Hilbert spaces if and only if u is invertible and $u^{-1} = u^*$.*

Proof. We have that

$$\|u\xi\|^2 = \langle u\xi, u\xi \rangle = \langle u^* u\xi, \xi \rangle = \langle \xi, \xi \rangle$$

for all $\xi \in H$ if and only if $u^* u = 1$. Since u is invertible, $u^* = u^{-1}$. □

Proposition 1.2. *Let $A, B \in \mathcal{B}(H, K)$, and let $C \in \mathcal{B}(K, K')$.*

1. $(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$.
2. $(CA)^* = A^* C^*$.
3. *If $H = K$ (so $A \in \mathcal{B}(H)$), then $(A^*)^* = A$.*
4. *If A is invertible, then A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.*

Proposition 1.3. *If $A \in \mathcal{B}(H)$, $\|A^*\| = \|A\| = \|A^* A\|^{1/2}$.*

Remark 1.2. The second equality is something you don't get in Banach spaces.

Proof.

$$\begin{aligned} \|A\|^2 &= \sup_{\xi \in (H)_1} \langle A\xi, A\xi \rangle = \sup_{\xi \in (H)_1} \langle A^* A\xi, \xi \rangle \\ &\leq \sup_{\xi \in (H)_1} \langle A^* A\xi, \xi \rangle \leq \sup_{\xi \in (H)_1} \|A^* A\xi\| \|\xi\| \\ &= \|A^* A\| \leq \|A^*\| \|A\|. \end{aligned}$$

So we get that $\|A\| \leq \|A^*\|$. In particular, this holds for $\|A^*\|$, so we get $\|A^*\| \leq \|A\|$. Then all inequalities are equalities, so $\|A^*\| = \|A\| = \|A^* A\|^{1/2}$. □

Example 1.2. If $M_\varphi \in \mathcal{B}(L^2(X, \mu))$ with $\varphi \in L^\infty(X, \mu)$, is multiplication by φ, then $(M_\varphi)^* = M_{\overline{\varphi}}$.

Example 1.3. The right shift $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ given by $S(\alpha_1, \alpha_2, \dots) = (0, \alpha_1, \alpha_2, \dots)$ is isometric. Then $S^*(\alpha_1, \alpha_2, \dots) = (\alpha_2, \alpha_3, \dots)$.

1.2 Hermitian operators

Definition 1.3. $A \in \mathcal{B}(H)$ is **Hermitian** (or **self adjoint**) if $A = A^*$.

Proposition 1.4. *A is Hermitian if and only if $\langle A\xi, \xi \rangle \in \mathbb{R}$ for all $\xi \in H$.*

Proof. ($\implies$): We have

$$\langle A\xi, \xi \rangle = \langle \xi, A\xi \rangle = \overline{\langle A\xi, \xi \rangle},$$

so $\langle A\xi, \xi \rangle \in \mathbb{R}$.

($\Longleftarrow$): We would like to prove that if $\langle A\xi, \xi \rangle = \langle \xi, A\xi \rangle$ for all $\xi \in H$, then $\langle A\xi, \eta \rangle = \langle \xi, A\eta \rangle$ for all $\xi, \eta \in H$. We use a **polarization** trick: check that

$$\langle A\xi, \eta \rangle = \frac{1}{4} \sum_{i=0}^{3} i^k \left\langle A(\xi + i^k \eta), \xi + i^k \eta \right\rangle,$$

$$\langle \xi, A, \eta \rangle = \frac{1}{4} \sum_{i=0}^{3} i^k \left\langle \xi + i^k \eta, A(\xi + i^k \eta) \right\rangle.$$

The right hand sides are equal, so the left hand sides are, as well. □

Proposition 1.5. *Let $A \in \mathcal{B}(H)$.*

1. $\|A\| = \sup_{\xi, \eta \in (H)_1} |\langle A\xi, \eta \rangle|$.
2. *If $A = A^*$, then* $\|A\| = \sup_{\xi \in (H)_1} |\langle A\xi, \xi \rangle|$.

Proof. For (1), we have $\geq$. For $\leq$, take $\eta = \frac{A\xi}{\|A\xi\|}$ for ξ with $A\xi \neq 0$.

For (2), we use

$$\begin{aligned} \langle A(\xi \pm \eta), \xi \pm \eta \rangle &= \langle A\xi, \xi \rangle \pm \langle A\xi, \eta \rangle \pm \langle A\eta, \xi \rangle + \langle A\eta, \eta \rangle \\ &= \langle A\xi, \xi \rangle \pm \langle A\xi, \eta \rangle \pm \overline{\langle A\xi, \eta \rangle} + \langle A\eta, \eta \rangle \\ &= \langle A\xi, \xi \rangle \pm 2 \operatorname{Re} \langle A\xi, \eta \rangle + \langle A\eta, \eta \rangle \end{aligned}$$

By subtracting one from the other, we get

$$\begin{aligned} 4 \operatorname{Re} \langle A\xi, \eta \rangle &= \langle A(\xi + \eta), \xi + \eta \rangle - \langle A(\xi - \eta), \xi - eta \rangle \\ &\leq \left(\sup_{\xi \in (H)_1} |\langle A\xi, \xi \rangle| \right) (\|\xi + \eta\|^2 + \|\eta - \eta\|^2) \\ &= 2 \left(\sup_{\xi \in (H)_1} |\langle A\xi, \xi \rangle| \right) (\|\xi\|^2 + \|\eta\|^2) \\ &\leq 4 \sup_{\xi \in (H)_1} |\langle A\xi, \xi \rangle|. \end{aligned}$$

By part 1, we get $\|A\| \leq \sup_{\xi \in (H)_1} |\langle A\xi, \xi \rangle|$. □

Corollary 1.1. *If $\langle A\xi, \xi \rangle = 0$ for all $\xi \in H$, then $A = 0$.*

Proof. For any $A \in \mathcal{B}(H)$, we can decompose A as two self-adjoint operators:

$$A = \frac{A + A^*}{2} + \frac{A - A^*}{2i}.$$

If $\langle A\xi, \xi \rangle = 0$, then this is true for each of these two parts. So each of these parts has norm equal to 0 by the previous proposition. □